Tutorial 4 5-10-2016

<u>Topics</u>: ○ Integration of complex-valued function of real variable
○ Contour integral
○ Anti-derivative and path independence

Integration of complex-valued function of real variable

<u>Defn</u>: Let $w(t) = u(t) + iv(t)$ be a complex-valued function of real variable. The integral of $w(t)$ is defined by

$$\int w(t)\,dt = \int u(t)\,dt + i\int v(t)\,dt.$$

<u>Remark</u>: The Fundamental theorem of Calculus also holds in this case.
i.e. If $W'(t) = w(t)$, then

$$\int_a^b w(t)\,dt = \int_a^b W'(t)\,dt = W(b) - W(a).$$

<u>Example</u>: 1) Show that for non-zero integer n,

$$\int_0^\pi e^{(1+ni)x}\,dx = \int_0^\pi e^x \cos nx\,dx + i\int_0^\pi e^x \sin nx\,dx.$$

Hence find the integrals on the R.H.S.

<u>Ans</u>: Note that

$$\int_0^\pi e^{(1+ni)x}\,dx = \int_0^\pi e^x \cdot e^{inx}\,dx$$

$$= \int_0^\pi e^x (\cos nx + i\sin nx)\,dx$$

$$= \int_0^\pi e^x \cos nx\,dx + i\int_0^\pi e^x \sin nx\,dx. \quad \text{---}(*)$$

Also we have

$$\int_0^\pi e^{(1+ni)x}\,dx = \left.\frac{e^{(1+ni)x}}{(1+ni)}\right|_0^\pi$$

$$= \frac{1}{1+ni}\left(e^{\pi + in\pi} - 1\right)$$

$$= \frac{1}{1+ni}\left((-1)^n e^{\pi} - 1\right)$$

$$= \frac{1-ni}{1+n^2}\left((-1)^n e^{\pi} - 1\right)$$

$$= \frac{1}{1+n^2}\left((-1)^n e^{\pi} - 1\right) + i\left(\frac{-n}{1+n^2}\left((-1)^n e^{\pi} - 1\right)\right)$$

By comparing the real and imaginary parts on both sides of $(*)$, we have

$$\int_0^{\pi} e^x \cos nx \, dx = \frac{1}{1+n^2}\left((-1)^n e^{\pi} - 1\right) \quad \text{and}$$

$$\int_0^{\pi} e^x \sin nx \, dx = \frac{-n}{1+n^2}\left((-1)^n e^{\pi} - 1\right)$$

Remark: Try to compute $\int_0^{\pi} e^{mx} \cos nx \, dx$ and $\int_0^{\pi} e^{mx} \sin nx \, dx$

by the same trick.

Contour Integral

Defn: Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise differentiable arc.

- γ is said to be closed if $\gamma(a) = \gamma(b)$
- γ is said to be simple if γ is injective on (a,b).

Defn: Let $\gamma : [a,b] \to \mathbb{C}$ be a differentiable arc.
Let $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$ be an cpl-valued function and $\gamma([a,b]) \subseteq \Omega$.
The contour integral of f along γ is defined to be

$$\int_{\gamma} f \, dz = \int_a^b f(\gamma(t))\, \gamma'(t) \, dt.$$

(complex multiplication, not dot product

Example: 1) Compute the contour integral of $f(z) = z^2$ along

a) $\gamma_1(t) = ti$, $t \in [0,1]$

b) $\gamma_2(t) = t + i$, $t \in [0,1]$

c) $\gamma_3(t) = t + t^2 i$, $t \in [0,1]$.

Ans: a)
$$\int_{\gamma_1} f(z)dz = \int_0^1 (ti)^2 (i)\, dt = -i\int_0^1 t^2 dt = \frac{-i}{3}$$

b)
$$\int_{\gamma_2} f(z)dz = \int_0^1 (t+i)^2 (1) dt = \int_0^1 (t^2 + 2it - 1) dt = \int_0^1 (t^2-1) dt + 2i \int_0^1 t\, dt = \frac{-2}{3} + i$$

c)
$$\int_{\gamma_3} f(z) dz = \int_0^1 (t + t^2 i)^2 (1 + 2ti) dt = \int_0^1 (t^2 + 2t^3 i - t^4)(1+2ti) dt$$
$$= \int_0^1 (t^2 - t^4 - 4t^4) dt + i \int_0^1 (2t^3 + 2t^3 - 2t^5) dt$$
$$= \frac{-2}{3} + \frac{2}{3} i$$

Remark: Note that we have
$$\int_{\gamma_1 + \gamma_2} f(z) dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z) dz = \int_{\gamma_3} f(z) dz.$$

2) Compute the contour integral where f is the principal branch of z^{ni} $(n \neq 0)$ & C is the semicircle with $z = e^{i\theta}$, $(0 \leq \theta \leq \pi)$.

Ans: $z^{ni} = \exp(ni \operatorname{Log} z)$

$= \exp(ni(\ln|z| + i \operatorname{Arg} z))$

So
$$\int_\gamma f(z)dz = \int_0^\pi \exp(ni(\ln(1) + i\theta)) \cdot ie^{i\theta} d\theta$$
$$= i\int_0^\pi e^{-n\theta} \cdot e^{i\theta} d\theta$$
$$= i \int_0^\pi e^{(-n+i)\theta} d\theta$$
$$= i \left. \frac{e^{(-n+i)\theta}}{-n+i} \right|_0^\pi$$
$$= \frac{i}{-n+i}\left(e^{(-n+i)\pi} - 1\right)$$
$$= \frac{i(-n-i)}{n^2+1}\left(-e^{-n\pi} - 1\right)$$
$$= -\frac{e^{-n\pi}+1}{1+n^2}(1-ni)$$

Anti-derivative and path independence

Prop: (Chain rule for analytic function)

If f is analytic and $z(t) = (x(t), y(t))$ is a differentiable curve sitting inside the domain of f, then we have
$$\frac{d}{dt} f(z(t)) = f'(z(t))\, z'(t)$$

Proof: let $f = u + iv$.

By Chain rule, we have
$$\frac{d}{dt} f(z(t)) = \frac{d}{dt} u(z(t)) + i \frac{d}{dt} v(z(t))$$

$$= \frac{d}{dt} u(x(t), y(t)) + i \frac{d}{dt} v(x(t), y(t))$$

$$= u_x x' + u_y y' + i(v_x x' + v_y y')$$

$$\overset{\text{CR-eqtn}}{=} u_x x' - v_x y' + i(v_x x' + u_x y')$$

$$= (u_x + i v_x)(x' + i y')$$

$$= f'(z(t))\, z'(t).$$

$\square$

Now suppose $f = F'$ for some analytic $F = U + iV$

Then $\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt$

$$= \int_a^b F'(\gamma(t))\gamma'(t)\,dt$$

$$= \int_a^b \frac{d}{dt} F(\gamma(t))\,dt$$

$$= \int_a^b \frac{d}{dt} U(\gamma(t)) + i \int_a^b \frac{d}{dt} V(\gamma(t))\,dt$$

$$= U(\gamma(b)) - U(\gamma(a)) + i(V(\gamma(b)) - V(\gamma(a)))$$

$$= F(\gamma(b)) - F(\gamma(a)).$$

Hence we have

Anti-derivative exists $\Longrightarrow$ path independence

i.e. $\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$

whenever $\gamma_1(a) = \gamma_2(a)$ and $\gamma_1(b) = \gamma_2(b)$